Inverse function of exponential function

Class of specific mathematical functions This article is about the function f(x) = ex and its generalizations of the form f(x) = abx. For functions of the form f(x,y) = xy, see Exponentiation. For functions of the form f(x) = xr, see Power function. The natural exponential function y = ex Exponential functions with bases 2 and 1/2 This article's lead section may be too long for the length of the article. Please help by moving some material from it into the body of the article. Please read the layout guide and lead section guidelines to ensure the section will still be inclusive of all essential details. Please discuss this issue on the article's talk page. (June 2021) In mathematics, the exponential function is the function f (x) = e x , {\displaystyle f(x)=e^{x},} where e = 2.71828... is Euler's constant. More generally, an exponential function is a function of the form f (x) = a b x , {\displaystyle f(x)=ab^{x},} where b is a positive real number, and the argument x occurs as an exponent. For real numbers c and d, a function of the form f (x) = a b c x + d {\displaystyle f(x)=ab^{cx+d}} is also an exponential function, since it can be rewritten as a b c x + d = (a b d) (b c) x . {\displaystyle ab^{cx+d}=\left(ab^{d}\right)\left(b^{c}\right)^{x}.} The exponential function f (x) = e x {\displaystyle f(x)=e^{x}} is sometimes called the natural exponential function for distinguishing it from the other exponential functions. The study of any exponential function can easily be reduced to that of the natural exponential function, since a b x = a e x ln b {\displaystyle ab^{x}=ae^{x\ln b}} As functions of a real variable, exponential functions are uniquely characterized by the fact that the growth rate of such a function (that is, its derivative) is directly proportional to the value of the function. The constant of proportionality of this relationship is the natural logarithm of the base b: d d x b x = b x log e b . {\displaystyle {\frac {d}{dx}}b^{x}=b^{x}\log _{e}b.} For b > 1, the function b x {\displaystyle b^{x}} is increasing (as depicted for b = e and b = 2), because log e b > 0 {\displaystyle \log _{e}b>0} makes the derivative always positive; while for b < 1, the function is decreasing (as depicted for b = 1/2); and for b = 1 the function is constant. The constant e = 2.71828... is the unique base for which the constant of proportionality is 1, so that the function is its own derivative: d d x e x = e x log e e = e x . {\displaystyle {\frac {d}{dx}}e^{x}=e^{x}\log _{e}e=e^{x}.} This function, also denoted as exp x, is called the "natural exponential function",[1][2][3] or simply "the exponential function". Since any exponential function can be written in terms of the natural exponential as b x = e x log e b {\displaystyle b^{x}=e^{x\log _{e}b}} , it is computationally and conceptually convenient to reduce the study of exponential functions to this particular one. The natural exponential is hence denoted by x ↦ e x {\displaystyle x\mapsto e^{x}} or x ↦ exp x . {\displaystyle x\mapsto \exp x.} The former notation is commonly used for simpler exponents, while the latter is preferred when the exponent is a complicated expression. The graph of y = e x {\displaystyle y=e^{x}} is upward-sloping, and increases faster as x increases.[4] The graph always lies above the x-axis, but becomes arbitrarily close to it for large negative x; thus, the x-axis is a horizontal asymptote. The equation d d x e x = e x {\displaystyle {\tfrac {d}{dx}}e^{x}=e^{x}} means that the slope of the tangent to the graph at each point is equal to its y-coordinate at that point. Its inverse function is the natural logarithm, denoted log , {\displaystyle \log ,} [nb 1] ln , {\displaystyle \ln ,} [nb 2] or log e ; {\displaystyle \log _{e};} because of this, some old texts[5] refer to the exponential function as the antilogarithm. The exponential function satisfies the fundamental multiplicative identity (which can be extended to complex-valued exponents as well): e x + y = e x e y for all x , y ∈ R . {\displaystyle e^{x+y}=e^{x}e^{y}{\text{ for all }}x,y\in \mathbb {R} .} It can be shown that every continuous, nonzero solution of the functional equation f (x + y) = f (x) f (y) {\displaystyle f(x+y)=f(x)f(y)} is an exponential function, f : R → R , x ↦ b x , {\displaystyle f:\mathbb {R} \to \mathbb {R} ,\ x\mapsto b^{x},} with b ≠ 0. {\displaystyle beq 0.} The multiplicative identity, along with the definition e = e 1 {\displaystyle e=e^{1}} , shows that e n = e × ⋯ × e n factors {\displaystyle e^{n}=\underbrace {e\times \cdots \times e} _{n{\text{ factors}}}} for positive integers n, relating the exponential function to the elementary notion of exponentiation. The argument of the exponential function can be any real or complex number, or even an entirely different kind of mathematical object (for example, a square matrix). The ubiquitous occurrence of the exponential function in pure and applied mathematics has led mathematician W. Rudin to opine that the exponential function is "the most important function in mathematics".[6] In applied settings, exponential functions model a relationship in which a constant change in the independent variable gives the same proportional change (that is, percentage increase or decrease) in the dependent variable. This occurs widely in the natural and social sciences, as in a self-reproducing population, a fund accruing compound interest, or a growing body of manufacturing expertise. Thus, the exponential function also appears in a variety of contexts within physics, chemistry, engineering, mathematical biology, and economics. Part of a series of articles on themathematical constant e Properties Natural logarithm Exponential function Applications compound interest Euler's identity Euler's formula half-lives exponential growth and decay Defining e proof that e is irrational representations of e Lindemann–Weierstrass theorem People John Napier Leonhard Euler Related topics Schanuel's conjecture vte Formal definition Main article: Characterizations of the exponential function The exponential function (in blue), and the sum of the first n + 1 terms of its power series (in red). The real exponential function exp : R → R {\displaystyle \exp \colon \mathbb {R} \to \mathbb {R} } can be characterized in a variety of equivalent ways. It is commonly defined by the following power series:[6][7] exp x := ∑ k = 0 ∞ x k k ! = 1 + x + x 2 2 + x 3 6 + x 4 24 + ⋯ {\displaystyle \exp x:=\sum _{k=0}^{\infty }{\frac {x^{k}}{k!}}=1+x+{\frac {x^{2}}{2}}+{\frac {x^{3}}{6}}+{\frac {x^{4}}{24}}+\cdots } Since the radius of convergence of this power series is infinite, this definition is, in fact, applicable to all complex numbers z ∈ ℂ (see § Complex plane for the extension of exp x {\displaystyle \exp x} to the complex plane). The constant e can then be defined as e = exp 1 = ∑ k = 0 ∞ (1 / k !) . {\textstyle e=\exp 1=\sum _{k=0}^{\infty }(1/k!).} The term-by-term differentiation of this power series reveals that d d x exp x = exp x {\displaystyle {\frac {d}{dx}}\exp x=\exp x} for all real x, leading to another common characterization of exp x {\displaystyle \exp x} as the unique solution of the differential equation y ′ (x) = y (x) , {\displaystyle y'(x)=y(x),} satisfying the initial condition y (0) = 1. {\displaystyle y(0)=1.} Based on this characterization, the chain rule shows that its inverse function, the natural logarithm, satisfies d d y log e y = 1 / y {\displaystyle {\frac {d}{dy}}\log _{e}y=1/y} for y > 0 , {\displaystyle y>0,} or log e y = ∫ 1 y 1 t d t . {\textstyle \log _{e}y=\int _{1}^{y}{\frac {1}{t}}\,dt.} This relationship leads to a less common definition of the real exponential function exp x {\displaystyle \exp x} as the solution y {\displaystyle y} to the equation x = ∫ 1 y 1 t d t . {\displaystyle x=\int _{1}^{y}{\frac {1}{t}}\,dt.} By way of the binomial theorem and the power series definition, the exponential function can also be defined as the following limit:[8][7] exp x = lim n → ∞ (1 + x n) n . {\displaystyle \exp x=\lim _{n\to \infty }\left(1+{\frac {x}{n}}\right)^{n}.} Overview The red curve is the exponential function. The black horizontal lines show where it crosses the green vertical lines. The exponential function arises whenever a quantity grows or decays at a rate proportional to its current value. One such situation is continuously compounded interest, and in fact it was this observation that led Jacob Bernoulli in 1683[9] to the number lim n → ∞ (1 + 1 n) n {\displaystyle \lim _{n\to \infty }\left(1+{\frac {1}{n}}\right)^{n}} now known as e. Later, in 1697, Johann Bernoulli studied the calculus of the exponential function.[9] If a principal amount of 1 earns interest at an annual rate of x compounded monthly, then the interest earned each month is x/12 times the current value, so each month the total value is multiplied by (1 + x/12), and the value at the end of the year is (1 + x/12)12. If instead interest is compounded daily, this becomes (1 + x/365)365. Letting the number of time intervals per year grow without bound leads to the limit definition of the exponential function, exp x = lim n → ∞ (1 + x n) n {\displaystyle \exp x=\lim _{n\to \infty }\left(1+{\frac {x}{n}}\right)^{n}} first given by Leonhard Euler.[8] This is one of a number of characterizations of the exponential function; others involve series or differential equations. From any of these definitions it can be shown that the exponential function obeys the basic exponentiation identity, exp (x + y) = exp x ⋅ exp y {\displaystyle \exp(x+y)=\exp x\cdot \exp y} which justifies the notation ex for exp x. The derivative (rate of change) of the exponential function is the exponential function itself. More generally, a function with a rate of change proportional to the function itself (rather than equal to it) is expressible in terms of the exponential function. This function property leads to exponential growth or exponential decay. The exponential function extends to an entire function on the complex plane. Euler's formula relates its values at purely imaginary arguments to trigonometric functions. The exponential function also has analogues for which the argument is a matrix, or even an element of a Banach algebra or a Lie algebra. Derivatives and differential equations The derivative of the exponential function is equal to the value of the function. From any point P on the curve (blue), let a tangent line (red), and a vertical line (green) with height h be drawn, forming a right triangle with a base b on the x-axis. Since the slope of the red tangent line (the derivative) at P is equal to the ratio of the triangle's height to the triangle's base (rise over run), and the derivative is equal to the value of the function, h must be equal to the ratio of h to b. Therefore, the base b must always be 1. The importance of the exponential function in mathematics and the sciences stems mainly from its property as the unique function which is equal to its derivative and is equal to 1 when x = 0. That is, d d x e x = e x and e 0 = 1. {\displaystyle {\frac {d}{dx}}e^{x}=e^{x}\quad {\text{and}}\quad e^{0}=1.} Functions of the form cex for constant c are the only functions that are equal to their derivative (by the Picard–Lindelöf theorem). Other ways of saying the same thing include: The slope of the graph at any point is the height of the function at that point. The rate of increase of the function at x is equal to the value of the function at x. The function solves the differential equation y′ = y. exp is a fixed point of derivative as a functional. If a variable's growth or decay rate is proportional to its size—as is the case in unlimited population growth (see Malthusian catastrophe), continuously compounded interest, or radioactive decay—then the variable can be written as a constant times an exponential function of time. Explicitly for any real constant k, a function f: R → R satisfies f′ = kf if and only if f(x) = cekx for some constant c. The constant k is called the decay constant, disintegration constant,[10] rate constant,[11] or transformation constant.[12] Furthermore, for any differentiable function f(x), we find, by the chain rule: d d x e f (x) = f ′ (x) e f (x) . {\displaystyle {\frac {d}{dx}}e^{f(x)}=f'(x)e^{f(x)}.} Continued fractions for ex A continued fraction for ex can be obtained via an identity of Euler: e x = 1 + x 1 − x x + 2 − 2 x x + 3 − 3 x x + 4 − ⋱ {\displaystyle e^{x}=1+{\cfrac {x}{1-{\cfrac {x}{x+2-{\cfrac {2x}{x+3-{\cfrac {3x}{x+4-\ddots }}}}}}}}} The following generalized continued fraction for ez converges more quickly:[13] e z = 1 + 2 z 2 − z + z 2 6 + z 2 10 + z 2 14 + ⋱ {\displaystyle e^{z}=1+{\cfrac {2z}{2-z+{\cfrac {z^{2}}{6+{\cfrac {z^{2}}{10+{\cfrac {z^{2}}{14+\ddots }}}}}}}}} or, by applying the substitution z = x/y: e x y = 1 + 2 x 2 y − x + x 2 6 y + x 2 10 y + x 2 14 y + ⋱ {\displaystyle e^{\frac {x}{y}}=1+{\cfrac {2x}{2y-x+{\cfrac {x^{2}}{6y+{\cfrac {x^{2}}{10y+{\cfrac {x^{2}}{14y+\ddots }}}}}}}}} with a special case for z = 2: e 2 = 1 + 4 0 + 2 2 6 + 2 2 10 + 2 2 14 + ⋱ = 7 + 2 5 + 1 7 + 1 9 + 1 11 + ⋱ {\displaystyle e^{2}=1+{\cfrac {4}{0+{\cfrac {2^{2}}{6+{\cfrac {2^{2}}{10+{\cfrac {2^{2}}{14+\ddots \,}}}}}}}}=7+{\cfrac {2}{5+{\cfrac {1}{7+{\cfrac {1}{9+{\cfrac {1}{11+\ddots \,}}}}}}}}} This formula also converges, though more slowly, for z > 2. For example: e 3 = 1 + 6 − 1 + 3 2 6 + 3 2 10 + 3 2 14 + ⋱ = 13 + 54 7 + 9 14 + 9 18 + 9 22 + ⋱ {\displaystyle e^{3}=1+{\cfrac {6}{-1+{\cfrac {3^{2}}{6+{\cfrac {3^{2}}{10+{\cfrac {3^{2}}{14+\ddots \,}}}}}}}}=13+{\cfrac {54}{7+{\cfrac {9}{14+{\cfrac {9}{18+{\cfrac {9}{22+\ddots \,}}}}}}}}} Complex plane Exponential function on the complex plane. The transition from dark to light colors shows that the magnitude of the exponential function is increasing to the right. The periodic horizontal bands indicate that the exponential function is periodic in the imaginary part of its argument. As in the real case, the exponential function can be defined on the complex plane in several equivalent forms. The most common definition of the complex exponential function parallels the power series definition for real arguments, where the real variable is replaced by a complex one: exp z := ∑ k = 0 ∞ z k k ! {\displaystyle \exp z:=\sum _{k=0}^{\infty }{\frac {z^{k}}{k!}}} Alternatively, the complex exponential function may defined by modelling the limit definition for real arguments, but with the real variable replaced by a complex one: exp z := lim n → ∞ (1 + z n) n {\displaystyle \exp z:=\lim _{n\to \infty }\left(1+{\frac {z}{n}}\right)^{n}} For the power series definition, term-wise multiplication of two copies of this power series in the Cauchy sense, permitted by Mertens' theorem, shows that the defining multiplicative property of exponential functions continues to hold for all complex arguments: exp (w + z) = exp w exp z for all w , z ∈ C {\displaystyle \exp(w+z)=\exp w\exp z{\text{ for all }}w,z\in \mathbb {C} } The definition of the complex exponential function in turn leads to the appropriate definitions extending the trigonometric functions to complex arguments. In particular, when z = it (t real), the series definition yields the expansion exp (i t) = (1 − t 2 2 ! + t 4 4 ! − t 6 6 ! + ⋯) + i (t − t 3 3 ! + t 5 5 ! − t 7 7 ! + ⋯) . {\displaystyle \exp(it)=\left(1-{\frac {t^{2}}{2!}}+{\frac {t^{4}}{4!}}-{\frac {t^{6}}{6!}}+\cdots \right)+i\left(t-{\frac {t^{3}}{3!}}+{\frac {t^{5}}{5!}}-{\frac {t^{7}}{7!}}+\cdots \right).} In this expansion, the rearrangement of the terms into real and imaginary parts is justified by the absolute convergence of the series. The real and imaginary parts of the above expression in fact correspond to the series expansions of cos t and sin t, respectively. This correspondence provides motivation for defining cosine and sine for all complex arguments in terms of exp (± i z) {\displaystyle \exp(\pm iz)} and the equivalent power series:[14] cos z := exp (i z) + exp (− i z) 2 = ∑ k = 0 ∞ (− 1) k z 2 k (2 k) ! , and sin z := exp (i z) − exp (− i z) 2 i = ∑ k = 0 ∞ (− 1) k z 2 k + 1 (2 k + 1) ! for all z ∈ C . {\displaystyle {\begin{aligned}\cos z&:={\frac {\exp(iz)+\exp(-iz)}{2}}=\sum _{k=0}^{\infty }(-1)^{k}{\frac {z^{2k}}{(2k)!}},\quad {\text{and}}\\\sin z&:={\frac {\exp(iz)-\exp(-iz)}{2i}}=\sum _{k=0}^{\infty }(-1)^{k}{\frac {z^{2k+1}}{(2k+1)!}}\end{aligned}}{\text{for all }}z\in \mathbb {C} .} The functions exp, cos, and sin so defined have infinite radii of convergence by the ratio test and are therefore entire functions (that is, holomorphic on C {\displaystyle \mathbb {C} }). The range of the exponential function is C ∖ { 0 } {\displaystyle \mathbb {C} \setminus \{0\}} , while the ranges of the complex sine and cosine functions are both C {\displaystyle \mathbb {C} } in its entirety, in accord with Picard's theorem, which asserts that the range of a nonconstant entire function is either all of C {\displaystyle \mathbb {C} } , or C {\displaystyle \mathbb {C} } excluding one lacunary value. These definitions for the exponential and trigonometric functions lead trivially to Euler's formula: exp (i z) = cos z + i sin z for all z ∈ C {\displaystyle \exp(iz)=\cos z+i\sin z{\text{ for all }}z\in \mathbb {C} } . We could alternatively define the complex exponential function based on this relationship. If z = x + iy, where x and y are both real, then we could define its exponential as exp z = exp (x + i y) := (exp x) (cos y + i sin y) {\displaystyle \exp z=\exp(x+iy):=(\exp x)(\cos y+i\sin y)} where exp, cos, and sin on the right-hand side of the definition sign are to be interpreted as functions of a real variable, previously defined by other means.[15] For t ∈ R {\displaystyle t\in \mathbb {R} } , the relationship exp (i t) ¯ = exp (− i t) {\displaystyle {\overline {\exp(it)}}=\exp(-it)} holds, so that | exp (i t) | = 1 {\displaystyle |\exp(it)|=1} for real t {\displaystyle t} and t ↦ exp (i t) {\displaystyle t\mapsto \exp(it)} maps the real line (mod 2π) to the unit circle in the complex plane. Moreover, going from t = 0 {\displaystyle t=0} to t = t 0 {\displaystyle t=t_{0}} , the curve defined by γ (t) = exp (i t) {\displaystyle \gamma (t)=\exp(it)} traces a segment of the unit circle of length ∫ 0 t 0 | γ ′ (t) | d t = ∫ 0 t 0 | i exp (i t) | d t = t 0 {\displaystyle \int _{0}^{t_{0}}|\gamma '(t)|dt=\int _{0}^{t_{0}}|i\exp(it)|dt=t_{0}} , starting from z = 1 in the complex plane and going counterclockwise. Based on these observations and the fact that the measure of an angle in radians is the arc length on the unit circle subtended by the angle, it is easy to see that, restricted to real arguments, the sine and cosine functions as defined above coincide with the sine and cosine functions as introduced in elementary mathematics via geometric notions. The complex exponential function is periodic with period 2πi and exp (z + 2 π i k) = exp z {\displaystyle \exp(z+2\pi ik)=\exp z} holds for all z ∈ C , k ∈ Z {\displaystyle z\in \mathbb {C} ,k\in \mathbb {Z} } . When its domain is extended from the real line to the complex plane, the exponential function retains the following properties: e z + w = e z e w e 0 = 1 e z ≠ 0 d d z e z = e z (e z) n = e n z , n ∈ Z for all w , z ∈ C {\displaystyle {\begin{aligned}e^{z+w}=e^{z}e^{w}\,\\e^{0}=1\,\\e^{z}eq 0\\{\tfrac {d}{dz}}e^{z}=e^{z}\\\left(e^{z}\right)^{n}=e^{nz},n\in \mathbb {Z} \end{aligned}}{\text{ for all }}w,z\in \mathbb {C} } . Extending the natural logarithm to complex arguments yields the complex logarithm log z, which is a multivalued function. We can then define a more general exponentiation: z w = e w log z {\displaystyle z^{w}=e^{w\log z}} for all complex numbers z and w. This is also a multivalued function, even when z is real. This distinction is problematic, as the multivalued functions log z and zw are easily confused with their single-valued equivalents when substituting a real number for z. The rule about multiplying exponents for the case of positive real numbers must be modified in a multivalued context: (ez)w ≠ ezw, but rather (ez)w = e(z + 2niπ)w multivalued over integers n See failure of power and logarithm identities for more about problems with combining powers. The exponential function maps any line in the complex plane to a logarithmic spiral in the complex plane with the center at the origin. Two special cases exist: when the original line is parallel to the real axis, the resulting spiral never closes in on itself; when the original line is parallel to the imaginary axis, the resulting spiral is a circle of some radius. 3D-Plots of Real Part, Imaginary Part, and Modulus of the exponential function z = Re(ex + iy) z = Im(ex + iy) z = |ex + iy| Considering the complex exponential function as a function involving four real variables: v + i w = exp (x + i y) {\displaystyle v+iw=\exp(x+iy)} the graph of the exponential function is a two-dimensional surface curving through four dimensions. Starting with a color-coded portion of the x y {\displaystyle xy} domain, the following are depictions of the graph as variously projected into two or three dimensions. Graphs of the complex exponential function Checker board key: x > 0 : green {\displaystyle x>0:\;{\text{green}}} x < 0 : red {\displaystyle x 0 : yellow {\displaystyle y>0:\;{\text{yellow}}} y < 0 : blue {\displaystyle y